Math 250 2.7 Precise Definitions of Limits (epsilon-delta)

Objectives

1) Use absolute value to express distances
2) Give a function, a value $x = a$, a limit L, and a numerical value of epsilon ε, find a numerical value of delta δ from a graph if
   a. Function is symmetric about $x = a$
   b. Function is not symmetric about $x = a$
3) Learn the precise definition (using epsilon, delta, N) of
   a. Finite limit
   b. Infinite limit
4) Understand and use the Triangle Inequality
5) Write an epsilon-delta $(\varepsilon - \delta)$ proof for
   a. Finite limit
   b. Infinite limit

Recall: $\lim_{x\to a} f(x) = L$ means "as x approaches a (from either the left or the right), $f(x)$ approaches L", where L is a y-coordinate.

The distance from an arbitrary value x to a, regardless of whether $x < a$ or $x > a$, is $|x-a|$.

If $x \neq a$, then this distance is always positive, meaning $|x-a| > 0$ or. $0 < |x-a|$

Similarly, the distance from $f(x)$ to L is $|f(x) - L|$.

Definition of a (Finite) Limit

$\lim_{x\to a} f(x) = L$ means that for any value of ε, there is a δ so that $|f(x) - L| < \varepsilon$ whenever $|x-a| < \delta$

Definition of an Infinite Limit

$\lim_{x\to a} f(x) = \infty$ means that for any value of $N > 0$, there is a δ so that $f(x) > N$ whenever $|x-a| < \delta$

$\lim_{x\to a} f(x) = -\infty$ means that for any value of $N > 0$, there is a δ so that $f(x) < -N$ whenever $|x-a| < \delta$

Steps for Writing an $(\varepsilon - \delta)$ Proof

Step 1: Work backward, using N or ε, to find δ in terms of N or ε. Namely, start from $|f(x) - L| < \varepsilon$ or $f(x) > N$ or $f(x) < -N$, substitute algebra for $f(x)$, make algebraic changes until you have $|x-a| < \delta$, where δ is defined as an expression involving N or ε.

Step 2: Write the actual proof, starting with $|x-a| < \delta$, reversing the algebra of step 1 until $|f(x) - L| < \varepsilon$ or $f(x) > N$ or $f(x) < -N$ results.

Triangle Inequality can be helpful with the algebra: $|x+y| \leq |x| + |y|$

Practice and Examples

1) Consider the function $f(x)=\begin{cases}\frac{1}{2}x+\frac{7}{2} & x\neq 3\\ \textit{undefined} & x=3\end{cases}$

   a. Sketch the graph for $x>0$
   b. Find $\lim_{x\to 3} f(x)=L$ from the graph.
   c. If $\varepsilon=1$, find δ so that $|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$
   d. If $\varepsilon=.5$, find δ so that $|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

2) Consider the function $f(x)=x^3-6x^2+12x-5$. Use GC to graph to find $\lim_{x\to 2} f(x)=L$ and answer:

   a. If $\varepsilon=1$, find δ so that $|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$
   b. If $\varepsilon=.5$, find δ so that $|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

3) Consider the function $f(x)=\begin{cases}2x-1 & x<2\\ \frac{1}{2}x+2 & x>2\end{cases}$

   a. Sketch the graph for $x>0$ and find $\lim_{x\to 2} f(x)=L$ from the graph.
   b. If $\varepsilon=2$, find δ so that $|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$
   c. If $\varepsilon=1$, find δ so that $|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

4) Write a proof that $\lim_{x\to 4}(4x-15)=1$

5) Write a proof that $\lim_{x\to 2}\frac{1}{(x-2)^2}=\infty$

Math 250 Briggs 2.7 Precise Definitions of Limits

Objectives

1) Use absolute value to express distances
2) Given a function, a value $x=a$, a limit L, and a numerical value of ϵ (epsilon), find a numerical value of δ (delta) from a graph if
   - function is symmetric about $x=a$
   - function is not symmetric about $x=a$.
3) Learn the precise definition of
   - a finite limit
   - an infinite limit.
4) Understand and use the Triangle Inequality
5) Write an ϵ-δ proof for
   - a finite limit
   - an infinite limit

Greek letters: epsilon ϵ (always rounded)
delta δ
lowercase. Δ is uppercase.

ϵ-δ proofs are the fully precise way to write proofs -- if you are a math major, you <u>will</u> have to write these proofs as part of a real analysis class.

ϵ is a number — very small, always positive, which refers to distances on the y-axis.

δ is a number - very small, always positive, which refers to distances on the x-axis.

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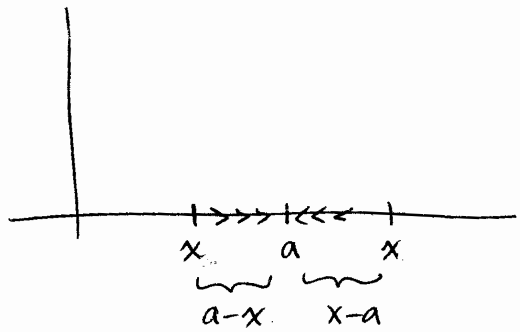
m250 2.7

Recall:

$$\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$$

means "as x approaches a (from either the right or the left, f(x) approaches L (y-coordinate)."



we want a single notation for the distance from x to a, regardless of whether x is left or right of a.

$|x-a|$ = distance from x to a

Ex: $x=5,\ a=3$: $x-a = |x-a| = |5-3| = 2$

Ex: $a=5,\ x=3$: $a-x = |a-x| = |5-3| = 2 = |x-a|$

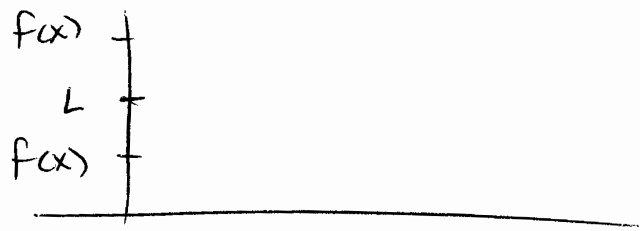
$|x-a| = |a-x|$ always

Note: If $x \neq a$, then $x-a>0$ or $a-x>0$

$|x-a|>0$.

or $0<|x-a|$.

Meanwhile on the y-axis, $f(x)$ is a y-coordinate which might be above or below L.



$|f(x) - L| =$ distance from $f(x)$ to L

$|f(x) - L| = |L - f(x)|$ always.

Precise Definition of Limit (finite) L

$\lim_{x \to a} f(x) = L$ means for any value of ϵ, there is a δ so that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$

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① Consider the function

$$f(x) = \begin{cases} \frac{1}{2}x + \frac{7}{2} & x \neq 3 \\ \text{undefined} & x = 3 \end{cases}$$ and $\lim_{x \to 3} f(x)$.

a) Sketch the graph for $x > 0$

b) Find $\lim_{x \to 3} f(x)$ from the graph. Call this limit L.

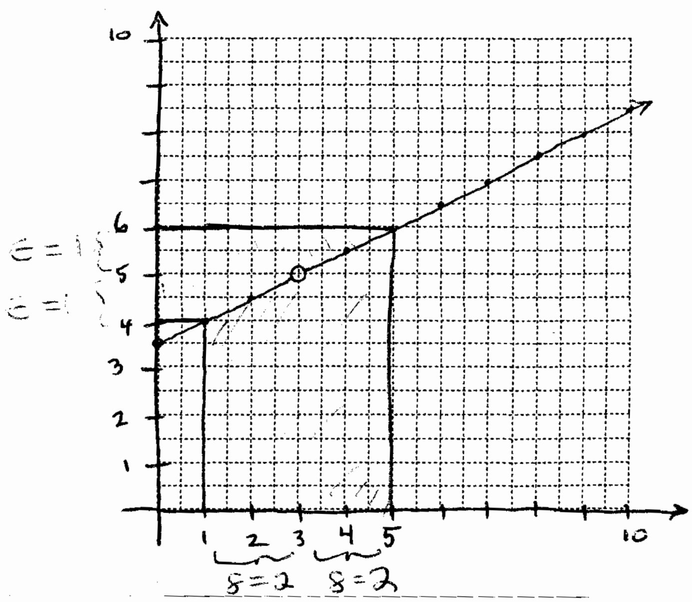
c) If $\epsilon = 1$, find δ so that

$|f(x) - L| < \epsilon$ whenever $0 < |x-3| < \delta$.

d) If $\epsilon = .5$, find δ so that

$|f(x) - L| < \epsilon$ whenever $0 < |x-3| < \delta$.

a)



b) $\lim_{x \to 3} f(x) = \boxed{5 = L}$

c) If $\epsilon = 1$, we consider y-values $|f(x) - 5| < 1$,
so the distance from 5 to $f(x)$ is at most 1 unit
We trace horizontal lines at $y = 5+1 \Rightarrow y = 6$
$y = 5-1 \Rightarrow y = 4$
until they meet the graph of $f(x)$. (green)
We draw vertical lines downward from these points
of intersection (red)

We see from the graph that these red lines are $x=1$ and $x=5$.

When $x=1$, the distance to $x=a=3$ is

$$|3-1|=2$$

Similarly, when $x=5$, the distance to $x=a=3$ is

$$|3-5|=|-2|=2.$$

So $\boxed{\delta=2}$ makes the statement true:

$$|f(x)-5|<1 \quad \text{whenever} \quad 0<|x-3|<2$$

Note: We could also do this by algebra:

If $\epsilon=1$ and $L=5$, then $L+\epsilon=6$, $L-\epsilon=4$

Set $f(x)=6$ $\quad \frac{1}{2}x+\frac{7}{2}=6$

$\frac{1}{2}x=\frac{5}{2}$

$x=5 \longrightarrow |5-3|=2$

set $f(x)=4$ $\quad \frac{1}{2}x+\frac{7}{2}=4$

$\frac{1}{2}x=\frac{1}{2}$

$x=1 \longrightarrow |1-3|=2$

find the distance from these x-values to $x=a=3$.

$\boxed{\delta=2}$.

d) If $\epsilon=.5$, $\quad L+\epsilon=5+.5=5.5=\frac{11}{2}$

$L-\epsilon=5-.5=4.5=\frac{9}{2}$

Set $f(x)=\frac{11}{2}$ $\quad \frac{1}{2}x+\frac{7}{2}=\frac{11}{2}$

$\frac{1}{2}x=2$

$x=4 \longrightarrow |4-3|=1$

set $f(x)=\frac{9}{2}$ $\quad \frac{1}{2}x+\frac{7}{2}=\frac{9}{2}$

$\frac{1}{2}x=1$

$x=2 \longrightarrow |2-3|=1$

$\boxed{\delta=1}$

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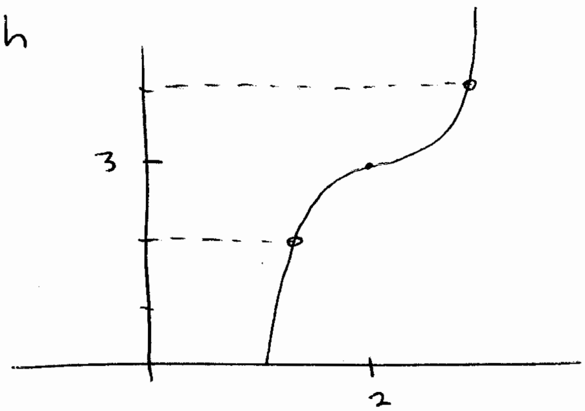
(2) Consider the function $f(x) = x^3 - 6x^2 + 12x - 5$ and $\lim_{x \to 2} f(x) = L$. Use GC to graph and answer.

a) If $\epsilon = 1$, find δ so that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x-2| < \delta.$$

b) If $\epsilon = \frac{1}{2}$, find δ so that ↑

Graph



$\boxed{Y=}$

$Y_1 = X\hat{}3 - 6X^2 + 12X - 5$

$f(2) = 2^3 - 6(2)^2 + 12(2) - 5$
$= 3$

f is a polynomial, continuous everywhere, so $\lim_{x \to 2} f(x) = f(2) = 3 = L$

$a = 2$.

If $\epsilon = 1$, we need horizontal lines $y = L + \epsilon = 3 + 1 = 4$
$y = L - \epsilon = 3 - 1 = 2$

We have the GC graph these lines and find points of intersection!

$Y_1 = \text{poly}$
$Y_2 = 4$

$\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$ = $\boxed{\text{CALC}}$ 5. Intersect
First curve? $\boxed{\text{enter}}$
Second curve? $\boxed{\text{enter}}$
Guess? $\boxed{\text{enter}}$

$f(x) = 4$ when $x = 3$. $\Rightarrow |x - a| = |3 - 2| = 1$

Repeat for $Y_2 = 2$

$f(x) = 2$ when $x = 1$ $\Rightarrow |x - a| = |1 - 2| = 1$

So:

$$|f(x) - 3| < 1 \quad \text{whenever} \quad |x - 2| < 1$$

$$\boxed{\epsilon = \delta = 1}$$

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b) If $\epsilon = \frac{1}{2}$, then $L + \epsilon = 3 + \frac{1}{2} = \frac{7}{2}$ or 3.5

and $L - \epsilon = 3 - \frac{1}{2} = \frac{5}{2}$ or 2.5

Graph $\begin{cases} y_1 = \text{poly} \\ y_2 = 3.5 \end{cases}$ and solve for intersection

$f(x) = 3.5$ when $x \approx 2.7937005$

$\Rightarrow |x - a| = |2 - 2.7937005|$ OR $= |2.7937005 - 2|$

$\approx .7937005$

Repeat $y_2 = 2.5$

$f(x) = 2.5$ when $x \approx 1.2062995$

$\Rightarrow |x - a| = |2 - 1.2062995|$ OR $= |1.2062995 - 2|$

$\approx .7937005$

if $\epsilon = .5$, then $\boxed{\delta \approx .7937005}$

Q: If we round to $\delta = .8$, is the result valid, namely whenever $0 < |x - 2| < .8$, $|f(x) - 3| < \frac{1}{2}$?

<u>No!</u> if $x - 2 = .799$ which is less than .8 but not less than .7937005,

$x = 2.799$

$f(2.799) \approx 3.510082399 > 3 + \epsilon$ oops!

Q: If we chop $\delta = .79$, is the result valid? <u>YES!</u>

In both examples ① and ②, we got the same value of δ for both the result using $L + \epsilon$ and the result using $L - \epsilon$. What if we don't get the same result?

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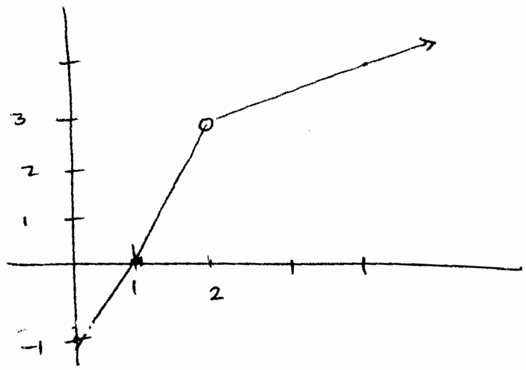
(3) Consider $g(x) = \begin{cases} 2x-1 & x<2 \\ \frac{1}{2}x+2 & x>2 \end{cases}$

a) Graph and find $\lim_{x\to 2} g(x) = L$



$\lim_{x\to 2} g(x) = \boxed{3} = L$

$\lim_{x\to 2^-} 2(2)-1 = 3$

$\lim_{x\to 2^+} \frac{1}{2}(2)+2 = 3$

b) If $\epsilon=2$, find δ.
c) If $\epsilon=1$, find δ.

b) If $\epsilon=2$, $L+\epsilon = 3+2=5$

$g(x)=5$ uses $\frac{1}{2}x+2=5$

$\frac{1}{2}x=3$

$x=6 \Rightarrow |x-a|$
$=|6-2|=4$

$L-\epsilon = 3-2=1$

$g(x)=1$ uses $2x-1=1$

$2x=2$

$x=1 \Rightarrow |x-a|$
$=|1-2|=1$

If we use $\delta=4$, $|x-a|$ for $x<a$ will make $|f(x)-L|>\epsilon$ ☹.

Choose δ = minimum of the two values

δ = minimum $(1,4)$

$\boxed{\delta=1}$ when $\epsilon=2$.

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c) If $\epsilon = 1$ $L + \epsilon = 3 + 1 = 4$

$g(x) = 4$ uses $\frac{1}{2}x + 2 = 4$

$\frac{1}{2}x = 2$

$x = 4 \Rightarrow |x - a|$

$= |4 - 2| = 2$

$L - \epsilon = 3 - 1 = 2$

$g(x) = 2$ uses $2x - 1 = 2$

$2x = 3$

$x = 1.5 \Rightarrow |x - a|$

$\Rightarrow |1.5 - 2| = .5$

$\delta = \min(.5, 2)$

$\boxed{\delta = .5}$

But what if $\lim_{x \to a} f(x) = +\infty$ or $-\infty$?

- we cannot do arithmetic w/ ∞ or $-\infty$, so $L + \epsilon$ and $L - \epsilon$ are nonsense.

INFINITE LIMITS

Defn: $\lim_{x \to a} f(x) = \infty$ means that for any positive # N, there exists a corresponding δ so that

$f(x) > N$ whenever $0 < |x - a| < \delta$.

Defn $\lim_{x \to a} f(x) = -\infty$ means that for any positive # N, there exists a corresponding δ so that

$f(x) < -N$ whenever $0 < |x - a| < \delta$.

We don't use ϵ (very small # added + subtracted from L) instead we use a big number N which f(x) must exceed.

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To write an actual proof of a limit L, we do not use numerical values for ϵ and δ.

Instead we write δ in terms of ϵ, that is, as an expression involving ϵ.

Steps for Writing an δ-ϵ Proof

step 1: Work backward, using N or ϵ to find δ.

Start from $|f(x) - L| < \epsilon$ or $f(x) > N$

substitute algebra for $f(x)$.

make algebraic changes until you have

$$|x - a| < \text{stuff involving } \epsilon.$$

Call the stuff involving $\epsilon = \delta$

step 2: Write the actual proof, starting with

$|x - a| < \delta$

Reversing the algebra of step 1 until you end with

$$|f(x) - L| < \epsilon. \quad (\text{or } f(x) > N)$$

Sometimes we need the Triangle Inequality to do this algebra. For any x and y:

$$|x + y| \leq |x| + |y|$$

Try it:

if $x = 2, y = 7$

$|2 + 7| \leq |2| + |7|$

$9 \leq 9$ ✓

If $x = -2, y = 7$

$|-2 + 7| \leq |2| + |7|$

$5 \leq 2 + 7$ ✓

Other useful notes $|c \cdot x| = |c| \cdot |x|$

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(4) Write a proof that $\lim_{x \to 4} (4x-15) = 1$.

Step 1: work backward to find δ.

$$|f(x) - L| < \epsilon$$
$$|4x-15-1| < \epsilon$$
$$|4x-16| < \epsilon$$
$$|4(x-4)| < \epsilon$$
$$|4| \cdot |x-4| < \epsilon$$
$$4 \cdot |x-4| < \epsilon$$
$$|x-4| < \epsilon/4 \quad \leftarrow \text{define } \delta = \epsilon/4.$$

Step 2: Write proof

If $|x-4| < \delta$
$$|x-4| < \epsilon/4$$
$$4|x-4| < \epsilon$$
$$|4x-16| < \epsilon$$
$$|4x-15-1| < \epsilon$$
then $|f(x) - L| < \epsilon$, as desired.

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(5) Let $f(x) = \dfrac{1}{(x-2)^2}$. Prove that $\lim_{x \to 2} f(x) = \infty$.

<u>step 1</u>: Assume $N > 0$

$$f(x) > N$$

$$\frac{1}{(x-2)^2} > N$$

$$1 > N \cdot (x-2)^2$$

$(x-2)^2$ is positive, so inequality unchanged

$$N(x-2)^2 < 1$$

$$(x-2)^2 < \frac{1}{N}$$

ditto, $N > 0$.

$$|x-2| < \frac{\sqrt{1}}{\sqrt{N}}$$

for any value $c > 1$
if $c > x$ then $\sqrt{c} > \sqrt{x}$.
(If we've got big #s, we can take $\sqrt{\ }$ both sides
$\sqrt{x^2} = |x|$
but we need absolute values)

$$|x-2| < \frac{1}{\sqrt{N}} = \delta$$

<u>step 2</u>: If $|x-2| < \delta$

$$|x-2| < \frac{1}{\sqrt{N}}$$

$$(x-2)^2 < \frac{1}{N}$$

$$N(x-2)^2 < 1$$

$$N < \frac{1}{(x-2)^2}$$

$$\frac{1}{(x-2)^2} > N$$

then $f(x) > N$. Hurray!